Analysis of Fractals from a Mathematical and Real-World Perspective

Anup A. Somalwar
Mathematics

Historical Background

Fractals give us a way to visually and quantitatively observe various abstract and real-world phenomena. Mathematician Benoit Mandelbrot coined the word "fractal" in 1975 (Stewart, Clark, Mandelbrot, et al., 5) while working at IBM but other mathematicians had studied these geometric structures long before him. During the 18th and 19th centuries, Calculus was gaining mass appeal. Most mathematicians of this period believed that all continuous curves could have tangent lines drawn at any point on the function, except at cusps. In 1872, Mathematician Karl Weierstrass, described a family of curves that were continuous but not differentiable at any point (Stewart, Clark, Mandelbrot, et al., 6-7). This discovery surprised mathematicians because it challenged current ideas about mathematics (Eglash, 12). Mathematicians were somewhat uncomfortable with the idea of continuous curves that could repeat, or iterate, themselves infinite times within a finite space. Two questions arose from this period: What are the dimensions of these curves, and can we measure the lengths of these "pathological curves" as they iterate over and over again in a finite space?

In the early 20th century, Mathematician Felix Hausdorff, helped make the idea of fractional dimension clear, believing fractals to be in an in-between dimension as they were not quite two-dimensional objects or one-dimensional mathematical structures. His main contribution to this area of mathematics was the Hausdorff dimension (Barnsley, p. 171):

$$d = \frac{\ln N}{\ln s}$$

where d is the dimension, N is the number of self-similar pieces of the geometric object in question, and s is the scaling or magnification factor.

Mandelbrot took Hausdorff's idea one-step further when he explored how to measure the coastline of Great Britain using different measures. He determined that if you use a smaller and smaller measuring instrument approaching a length of zero units, the approximation of the coast gets larger and larger, approaching infinity. He also discusses the idea of fractional dimension, comparing the coastline of England to the Koch Snowflake, yielding a dimension of 1.26 for the Koch Snowflake and 1.25 for the coastline (Mandelbrot, 1-5). He was not implying that the Koch Snowflake and the coastline of England had the same structure but that the overall dimension, or statistical self-similarity, was similar. This historical background lays the groundwork for our study of the mathematics underlying fractals and their applications to our world.

Definitions and Classic Examples of Fractals

A fractal is "a rough or fragmented geometric shape that can be split into parts, each of which is a reduced-size copy of the whole" (Frailey, "Fractals"). Four main mathematical characteristics define fractals: recursion, self-similarity, fractional or non-integer dimension, and the concept of infinity (Eglash, 17).

1. **Recursion** refers to an algorithmic reliance on the previous iteration of a fractal to produce the next iteration of the fractal design. A recursive example that may be familiar to readers is the Fibonacci sequence: 1,1, 2, 3, 5, 8… which can be defined as $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$, where f_n is the n^{th} term of the sequence, and f_{n-1} and f_{n-2} are the two previous terms.
2. **Self-similarity** refers to a fractal representing its original form at different levels of magnification; in layman's terms, whether you zoom in or out of the graphical representation of the fractal, the fractal maintains its original structure. Mandelbrot (p. 1) defined statistical self-similarity as "each portion can be considered a reduced scale image of the whole," referring to the scaling dimension.
3. **Fractional dimension**, calculated via the Hausdorff dimension definition, relates to the self-similarity characteristic of a fractal appearing the "same" despite the scaling factor.
4. Many fractals, such as the Cantor set, make use of the concept of **infinity** as they can iterate themselves an infinite number of times within a finite space.

Fractals are examples of dynamical systems, so it is important to review some important definitions and terms:

1. A **dynamical system**, in very simplistic terms, describes how a set of variables behaves under different states or conditions over time (Devaney, 2-7).
2. **Chaos** describes the phenomenon that occurs when slight changes to initial conditions, in the dynamical system, result in varied behavior for the entire system (Devaney, 49-50). A famous example is the Lorenz attractor. This result was produced after Dr. Edward Lorenz was trying to make weather predictions using differential equations and found you could not make accurate long-term predictions due to the chaotic nature of the system.
3. When polynomials and trigonometric functions are iterated, they produce simple dynamical systems. The dynamics of these systems can be described by **orbits**, or set of iterated points, that may converge to **fixed point(s)** or **periodic points**, or may never converge, moving chaotically throughout the space.
4. Fixed points can behave as **attractors** or **repellers**. If a fixed point behaves as an attractor, nearby orbits of the dynamical system will converge to the value of the fixed point. However, on the other hand, if a fixed point behaves as a repeller, nearby orbits of the dynamical system will not converge to the value of the fixed point of interest (Devaney, 17-37).

Through the use of phase-portraits and cobweb diagrams, it is possible to qualitatively observe the behavior of these dynamical systems and make inferences regarding their behavior.

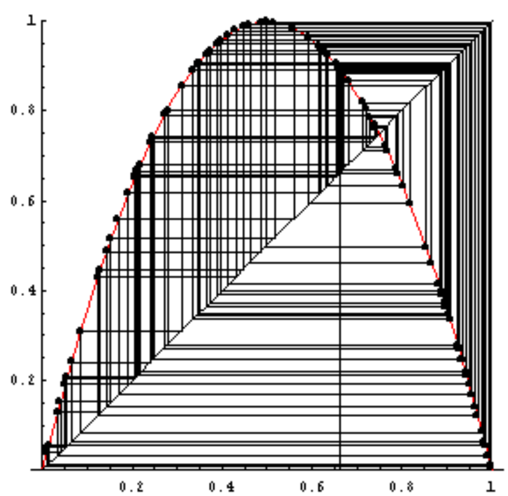

Figure 1 – Cobweb Diagram

The classic example of a fractal is known as the Cantor Set, created by Georg Cantor. He devised this fractal by taking a line segment of length 1 and proceeded to cut the line segment into pieces by removing the middle third of each line. In other words, you start out with line segment on the real number line on [0,1], then you break this down into two sub-interval $[0,\frac{1}{3}] \cup [\frac{2}{3},1]$, and continue iterating or repeating the process resulting in the union of infinite sets, each a subset of [0,1].

Another interesting result is what happens to the length of the set upon iteration, it starts out at 1, then $1-\frac{1}{3}=\frac{2}{3}$, followed by $\frac{2}{3}-\frac{1}{3}(\frac{2}{3})=\frac{2}{3}-\frac{2}{9}=\frac{4}{9}$, ... The length gets smaller each time. The geometric series $\frac{1}{3}\sum_{i=0}^{\infty}\left(\frac{2}{3}\right)^{in}$, describes the lengths of the all the segments being removed, and using the results from Calculus, we know this series converges to $\frac{aa}{1-rr}=\frac{\frac{1}{3}}{1-\frac{2}{3}}=\frac{\frac{1}{3}}{\frac{1}{3}}=1$, where *a* is the first term of the geometric sequence and *r* is the common ratio (Crilly, Earnshaw, Jones, 10). However, if the entire length of the interval is 1 and the convergence of all the parts removed is also 1, then 1 -1 = 0, meaning the length of the entire Cantor set is 0. If you continue this recursive process, the n^{th} iteration of this fractal will consist of an infinite number of points. These points will be disconnected from each other and eventually reach length zero on [0,1].

The dimension of the Cantor set can also be found using the Hausdorff Dimension definition, $d=\frac{\ln n}{\ln s}=\frac{\ln(2)}{\ln(3)}=0.63$ ("Dimension of Cantor Set"). Notice: There are 2^n divisions of the intervals and you are scaling down the Cantor set each time by a factor of 3 or one third.

Similarly, we can explore the behavior of the Koch Curve created by Helge von Koch. To create the Koch Curve, you start out with a line segment of with length of one unit. Instead of removing the middle-thirds of the line segment as you do in the Cantor Set, you actually add two segments of length one third of the original line. Another way to think about this is that you take the middle third of the original line segment and replace it with the top part of an equilateral triangle. This process is iterated over and over again to produce more intricate geometric design. You can clearly see that through the various iterations of the Koch Curve, since you are adding line segments of $(\frac{1}{3})^n$ during each iteration, the curve will get longer and longer over time.

One interesting result we can show is that although the Koch Curve has infinite length, the Koch Snowflake encompasses an area that is finite. The Koch Snowflake is a variation of the Koch Curve (Figure 2). We first start out by finding the area of an equilateral triangle, given to us by the formula A $= \frac{\sqrt{3}}{4}s^2$. When there are no iterations, we obtain the area of the original triangle which is given to us by $\frac{\sqrt{3}}{4}s^2$, then for the first iteration, each side of the three new triangles is cut into thirds, so we add on $3\frac{\sqrt{3}}{4}(\frac{s}{3})^2$ to the area previously obtained, and continue the process. Notice, there is a pattern regarding the new triangles added on to base triangle, 3, 12, 48, ... = 3* 4^{i-1}. Written out, the expanded area looks like:

$$\frac{\sqrt{3}}{4}s^2 + 3\frac{\sqrt{3}}{4}(\frac{s}{3})^2 + 12\frac{\sqrt{3}}{4}(\frac{s}{9})^2 + 48\frac{\sqrt{3}}{4}(\frac{s}{27})^2 + \ldots = \frac{\sqrt{3}}{4}s^2(1 + \frac{3}{9} + \frac{3*4}{9^2} + \frac{3*4^2}{9^3} + \ldots)$$

$$= \frac{\sqrt{3}}{4}s^2(1 + \sum_{i=1}^{\infty}\frac{3*4^{i-1}}{9^i}).$$

Now using the results from Calculus, we have a geometric series that converges to $\frac{a}{1-r} = \frac{\frac{3}{9}}{1-\frac{4}{9}} = \frac{\frac{3}{9}}{\frac{5}{9}} = \frac{3}{5}$ and since we shifted the series, $1 + \frac{3}{5} = \frac{8}{5}$. This implies that the area for the Koch Snowflake, whose construction begins with a triangle with side length *s* is equal to $\frac{\sqrt{3}}{4}s^2\left(\frac{8}{5}\right) = \frac{2\sqrt{3}}{5}s^2$, has a finite area (Riddle, "Area of the Koch Snowflake"). We check the Hausdorff Dimension to verify that the Koch Curve is a fractal: there are four self-similar pieces after the first iteration and the curve shrinks by a factor of one third or 3, $d = \frac{\ln 4}{\ln 3} = 1.26$, a fractional dimension!

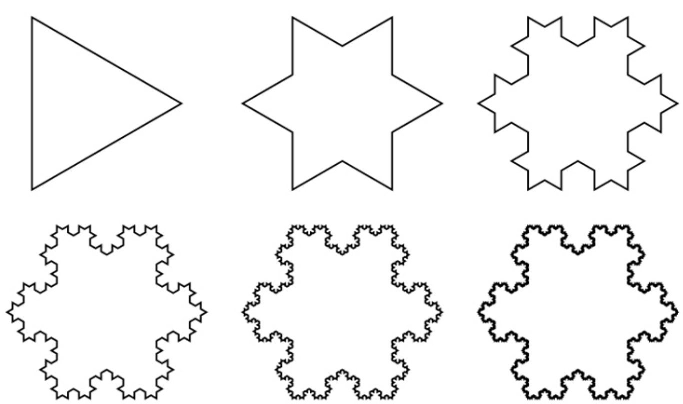

Figure 2 – Koch Snowflake

Another famous example of a fractal is known as Sierpiński's Triangle created by Wacław Sierpiński. In order to construct the fractal, first create a equilateral triangle. Then after identifying the midpoints of each side, connect those points to make another triangle and leave this triangle unshaded while shading in the other triangles that are formed. Then iterate the process a few times, always leaving the middle triangle in each larger triangle unshaded.

Notice, after the first iteration, there are four triangles, but one triangle of half-scale, or, an area of one-fourth the original triangle, is unshaded or removed. After the second iteration, there are three additional triangles at one-fourth scale, or one-sixteenth the area of the original triangle being removed (Figure 3), After the *n*th iteration, triangles of $\left(\frac{1}{4}\right)^n$ area will be removed; in other words, the size of the triangles being removed is getting smaller and smaller, eventually reaching an area of zero as *n* approaches infinity. You should also notice that there is a pattern with the numbers of triangles being removed from the original triangle: 1, 3, 9, 27... 3^n triangles will be removed upon *n* iterations. It appears that if this pattern continues, we will have an infinite number of triangles in a finite region, supporting the idea that this is an example of a fractal.

Combining the observations we have just made, we can create a geometric series that describes the area removed from the original triangle:

$$\frac{1}{4}\sum_{i=0}^{\infty}\left(\frac{1}{4}\right)^n * 3^n = \frac{1}{4}\sum_{i=0}^{\infty}\frac{1^n}{4^n} * 3^n = \frac{1}{4}\sum_{i=0}^{\infty}\frac{1}{4^n} * 3^n = \frac{1}{4}\sum_{i=0}^{\infty}\frac{3^n}{4^n}$$

$$= \frac{1}{4}\sum_{i=1}^{\infty}\left(\frac{3}{4}\right)^n \text{ (Crilly, Earnshaw, Jones, 14).}$$

Similar to the argument for the Cantor set, we can use results from Calculus to find where this geometric series will converge: $\frac{a}{1-r} = \frac{\frac{1}{4}}{1-\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{1}{4}} = 1$. This suggests the unshaded regions will eventually take over the space of the entire triangle over time. Finally, to absolutely make certain the Sierpiński triangle is a fractal, we need to identify the dimension. Using the Hausdorff Dimension definition, we see that there are three self-similar pieces of the triangle after the first iteration with a magnification scale of two (half scale). This gives us a fractional dimension of $\frac{\ln 3}{\ln 2} = 1.58$, supporting our argument regarding the fractal behavior of this geometric structure (Devaney, "The Sierpiński Triangle").

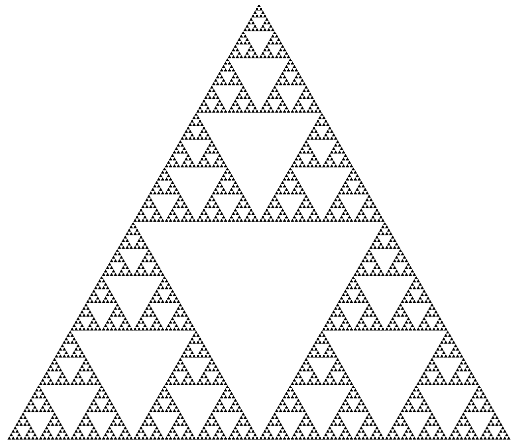

Figure 3 – Sierpiński's Triangle

Our next exploration of fractal geometry will involve discussion of the Julia and Mandelbrot sets. Both sets exist within the realm of the complex plane, whose subset is the real number line. **Complex numbers** are defined as $z = a+bi$, where a is a real number and the b is the imaginary component. The lower case i is notation for the square root of negative one, a number that does not exist in the set of real numbers. You can visualize a complex number on an *xy*-coordinate plane, where the *x*-coordinate represents the real number and the *y*-coordinate represents the imaginary component. The **Mandelbrot Set** is created by the rule: $z_{n+1} \rightarrow z_n^2 + c$, where z_n is a complex number, c is a complex number and z_{n+1} represents the iterated complex number. In order to use this formula, you take z_0, square the number and add c to it. This creates a new complex number, z_1. You then iterate the process *n*-number of times. Adding the complex number, c, to your original complex number, z_0, allows us to create different fractal sets. As noted above, each complex number can be plotted as a point so this deterministic rule generates many points in the plane after a few iterations. If you iterate this process, you will find that the new points either diverge to infinity or remain bounded for all time. The **Julia Set** is actually the boundary between the elements in the orbit that remain bounded and those that diverge to infinity. The Julia set can be created by iterating points using the same rule as the Mandelbrot set: $z_{n+1} = z_n^2 + c$. The Julia set is the set of starting values that remain bounded under iteration for a fixed value of c; the Mandelbrot set is the set of values of c for which the iteration remains bounded when starting at 0 (Crilly, Earnshaw, Jones, 22-23).

Computer models gave Mandelbrot, and later mathematicians, the ability to create rich visual representations of both sets. First, we will discuss how the computer can be used to generate Julia sets and then Mandelbrot sets. The computer recognizes each iteration of the rule, $z_{n+1} \rightarrow z_n^2 + c$, as a point or pixel and uses the distance formula to determine if the distance between the point and the origin is greater than or less than some pre-specified number. If the distance is larger than the pre-specified number, the point is outside the Julia set for that c

value. Similarly, if the distance is smaller than the pre-specified number, the point is inside the Julia set for that *c* value.

Computers have typically been programmed to produce different colors depending on the distances the iterated points are from the origin. Other color effects, also known as **escape value**, or how many iterations it took to end up outside a given distance from the origin, can produce even more visual effects. The *c* value used for calculations of the Julia set, will give your visual representation various forms. This gives us some qualitative support that this mathematical structure has chaotic properties. Similarly, the *c* value also determines if the Julia set is **connected** or not. A Julia set is connected if you can draw a line (or curve) between any two points in the set with the line remaining contained within the set. A Julia set is considered disconnected when you cannot do this; visually it appears to form a "dust" when the Julia set is disconnected. The Mandelbrot set is formed when the Julia set is connected (Crilly, Earnshaw, Jones, 22-23).

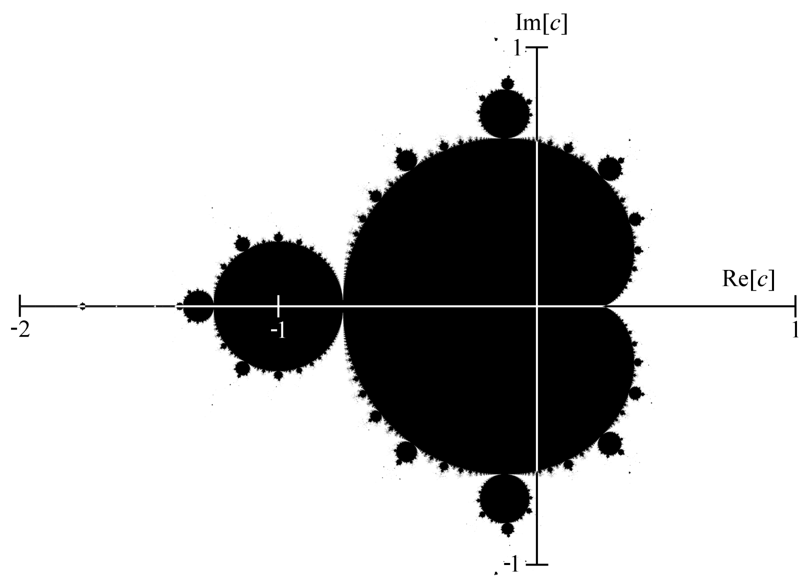

Figure 4 – Mandelbrot Set

In 1984, Mathematician Mitsuhiro Shishikura of the Tokyo Institute of Technology and the State University of New York at Stony Brook, proved that the boundary of the Mandelbrot set has dimension two using a proof involving the concept of bifurcation of parabolic periodic points (Mandelbrot, 111). A **bifurcation** occurs at a parameter value at which some variation induces dramatic changes within the dynamical system. An important idea used in the proof was that forming a union of all the connected Julia Sets could create Mandelbrot sets. Each connected Julia set represents a point in the Mandelbrot set. Shishikura shows the dimension of the Mandelbrot set is related to the dimension of the Julia sets that make it up. As the dimension of the Julia sets increases, so does the dimension of the Mandelbrot set, eventually reaching dimension two (Brown, "Mandelbrot Set is as complicated as it could be").

Applications of Fractals

Our exploration of the mathematics behind fractals has helped prepare the way for the second half of our journey: applications of fractals to the real world. You may be wondering how this knowledge of fractals could be useful to STEM professionals? Fractals have been useful in helping quantify and explain Brownian motion, applications in biology, the stock market, and used in various technologies. We will also explore how secondary mathematics teachers may introduce students to fractals, possibly piquing their overall interest in mathematics, and challenging some negative cultural stereotypes.

Brownian Motion

In 1827, Robert Brown discovered that particles of a substance, suspended in liquid, would continue to move. This phenomenon would later be known and studied as Brownian motion (Gordon, Clarke, Mandelbrot, et al., 11). In 1908, Einstein took his work further when he was studying how particles of a substance behave when exposed to heat and suspended in a liquid. He was able to use the mathematics of physics to quantify what he observed (kinetic-molecular theory of heat) in an experiment. Ultimately the findings of his experiment lead him to believe that the behavior of the particles could be predicted to some degree (Lee and Hoon, "Brownian Motion").

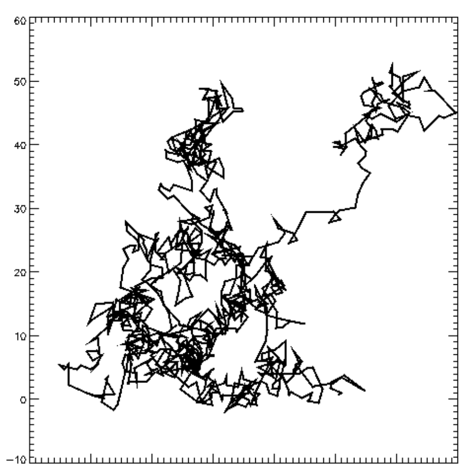

Figure 5 – Brownian Motion

Brownian motion of a particle, under a microscope, is said to have a fractional dimension of two. This is because if you try to predict the movement of a particular particle from one location to some other location under the microscope, you will find it will most likely have to take a path that fills the entire space before reaching the desired "stopping" point. Fractals are involved in fractional Brownian motion models; utilizing a **random walk** (a mathematical representation of the random behavior that occurs) and a **random iteration algorithm** (a deterministic rule that produces a "random" pattern), (Lee and Hoon, "Brownian Motion"). Fractal geometry and Brownian Motion have helped medical imaging technologists make more accurate diagnostic test readings through two primary avenues: classification and edge enhancement for ultrasonic liver images. **Classification** is when you use Brownian fractional motion to produce a motion vector that will accurately portray the surface of the liver. **Edge enhancement** is when you take each individual pixel from the current image of the liver and calculate its fractional dimension in comparison to the whole image. Doing these types of calculations helps the medical imaging technologists obtain a better picture of what is happening to the liver without an actual physical procedure—reducing unnecessary risk and time. Medical researchers Basu, Barba and Chan ("Texture Analysis in Cytology Using Fractals") show how fractional dimension can be used to try to distinguish between healthy and malignant cells after noticing that malignant cells consistently have a different fractional dimension compared to the healthy cells, independent of the type of cell: breast, bronchial, ovarian, and uterine.

Similarly, Smith, Lange, and Marks ("Fractal methods and results in cellular morphology...") give us insight into the methods in which biologists are using mathematical techniques to better understand cell morphology, relating to the overall physical characteristics of cells. There exist two major methods of calculating fractional dimension of these cells: using length (previously discussed with the rulers example in the introduction) or the mass of an object. The mass method involves "counting of border pixels in a sampling region, such as disc diameters, as a function of the sizes of the sample regions" when given a digital representation of the image (Smith, Lange, and Marks, "Fractal methods..."). The computer is then programmed to count the number of pixels contained in these round or box regions placed randomly all around the border of the image. Typically, the scientists will plot a diagram comparing the log of the pixels within each box vs. the log the measuring unit length. The power law describing this behavior can be represented mathematically as:

$\mu(r) = Ar^d$, where $\mu(r)$ is the number of pixels (considered mass), r is the length of the edge of the boxes or circle diameter, A is a pre-factor (a periodic function whose period is independent of the fractal dimension), D is the mass fractional dimension or slope of the plot of log $\mu(r)$ vs. log r.

(Smith, Lange, and Marks, "Fractal methods...").

Scientists must also account for lacunarity when trying to use fractals to measure the dimensions of cells. **Lacunarity** is a measure of "non-uniformity of structure or the degree of structural variance within an object;" a fractal structure that has high lacunarity may have large gaps or holes in its design and/or loses self-similarity when rotated 90 degrees (Smith, Lange, and Marks, "Fractal methods..."). Lacunarity is used to determine the differences in objects with similar fractional dimensions (Gould, Vadakkan, Poche, Dickinson, "Multifractal and Lacunarity Analysis of Microvascular Morphology and Remodeling"). We calculate lacunarity, or L, by calculating the mean and standard deviation of either the length or mass of the desired portion of the image, then normalizing those two values by dividing the variance by the square of the mean of all box or disc regions surveyed in the image (Smith, Lange, and Marks, "Fractal methods..."). A similar method to calculate L is dividing the standard deviation by the mean of all boxes or discs used, a statistical calculation, also known as the coefficient of variation.

Multifractals, or "objects whose fractal dimension varies as a function of location within in a set (image, frame)" (Smith, Lange, and Marks, "Fractal methods..."), were introduced to the author via the context of a biological application of fractal dimension but also have connections to the financial stock market as well. Previously, Brownian motion models utilizing a random walk and statistical distributions have been used to try to make accurate predictions about financial systems. However, real world events, such as the global stock market crisis of 2008, have revealed the flaws of using Brownian motion modeling systems, primarily with the inability to account for large spikes of sales or losses within the system within a short period of time (Clarke, Mandelbrot, Stewart, et. al., 127).

Financial Market Modeling using Fractals

Financial systems are good candidates to be modeled by fractals because the graphical representations of the rise and fall of stocks look similar on different scales when comparing price vs. time (idea of self-similarity). Mathematicians can replicate stock market prices using an initiator/generator model. In order to construct this model, first start out with the trend line called the initiator. Next, pick an interval of the graph you are interested in studying and apply the generator to that portion. The generator is a modification much like the replacement algorithm in the Koch curve in which each segment is replaced by a modified curve (with four segments in the case of the Koch curve). However, on a different portion of the same interval, you will iterate an inverted version of the generator. (Imagine the Koch substitution with the added triangular piece not symmetric. On one interval, substitute the original, and on the other, substitute the flipped version.) Note: the points on the generator must satisfy the conditions: $|dY_i| = (dt_i)^{1/2}$, where $|dY_i|$ is the height of some point (typically representing cost of the stock in a financial setting) and $(dt_i)^{1/2}$ is the distance that point travels on the t axis or time (Brennan, "Fractals and Financial Risk"). As you continue to break the interval up using more and more initiators, it produces a graph that more accurately represents the fluctuations than just a Brownian motion model, primarily due to the inverted generator. An even better representation of financial markets lies within a multi-fractal initiator/generator model according to Mandelbrot.

The multi-fractal model allows the financial analysts to incorporate the fact that many people will be buying stocks at different times and at different rates.
Budinski-Petkovic, Loncarevic, Jasksic and Vrhovac ("Fractal properties of financial markets") demonstrated that the U.S. S&P 500 financial bubbles in 1987, 2000, and 2007 could be well modeled by the Besicovitch-Ursell (B-U) fractal model. A **financial bubble** is a period of extreme growth (generally exponential in nature), sometimes followed by a cataclysmic drop in stock prices, or crash. One method econophysicists use to determine trends and make predictions about crashes is by fitting the financial data to the log-periodic power law using log-by-log plots.

The researchers in the paper first defined a saw-tooth, or tent map $f(x)$ and an another function, $g(x)$, to derive $h(x)$, a B-U function. The B-U function is a type of dynamical system that behaves as a fractal since it has a dimension of at most two and it has "almost periodicity"- behaving similarly but not exactly the same upon each iteration. The researchers used a non-linear fitting algorithm in MATLAB to show the B-U function produced relatively small regression errors. This gives further support to the argument Mandelbrot had made regarding the use of fractals in making financial predictions.

Practical Applications of Fractals in Technology

Cell phone technology also has benefited from this area of mathematics. In 1999, Dwight Jaggard and Douglas Werner found cell phone antennas arranged in a fractal pattern were superior in transmitting radio signals over other arrangements of the antenna (Stewart, Clark, Mandelbrot, et al., 21). Other researchers, Nathan Cohen and Robert Hohfeld, proved antenna designs that had self-similarity and symmetry properties (both of which are found in most fractals) performed well at various frequencies. Fractals are also useful in computational applications such as image and data compression, as well as data mining (Barnsley, Saupe, and Vrscay, 2-9).

The Internet also behaves in fractal-like ways when looking at how websites, search engines, and users interact with each other. The following observations help us see some patterns with information transfer across the Internet:

1. As a page's **connectedness** (linking of one page to another) increases, traffic to the website also increases.
2. The more traffic that a page generates, the greater its connections to other websites.
3. The greater a page's **rank** (where it appears on a search engine search), the greater its traffic.
4. The greater a page's rank, the greater its connectedness.
5. As the traffic increases, so does the page's rank.
6. Similarly, as the page increases its number of connections, the ranking of the page also increases (Stewart, Clark, Mandelbrot, et al., 93).

Expanding on these observations to form a type of system, you can see that the Internet can produce patterns that are self-organizing, and give the overall system some sense of predictability and order. Researchers have determined patterns of self-organization within hyperlinks, physical connections providing access to the internet, the information being exchanged on the internet, in the form of files, traffic patterns of data transmitted, and the ways in which people are accessing the data (Stewart, Clark, Mandelbrot, et al., 94). Computer scientists use a power-law distribution to study these types of phenomenon, similar to the techniques of the econophysicists. They discovered that when looking at data traffic, regardless of the time scale, self-similarity emerges. Most of the Internet traffic is small, checking an email, or browsing a website. However, this trend is often interrupted by spikes of high data transfers, such as when someone streams video on Netflix. Mathematically, the self-similarity and self-organizing results are due to the power law, which states, "the frequency of an item of size x is proportional to x^{-B}, where B is a constant" (Stewart, Clark, Mandelbrot, et al., 96). The power law, also known as the power law distribution, is called heavy-tailed because the right side of the distribution contains more data points than the other parts of the distribution (in comparison to the Gaussian or normal distribution).

Fractals in Education

In our final analysis of applications to the real world, we will discuss how secondary mathematics teachers might instruct students about fractals. Although fractals are not directly covered as a Common Core mathematics standard, basic concepts regarding fractals can be taught indirectly. Fractal investigations such as creating a Sierpiński triangle and identifying particular properties of the model relate to students' abilities to create mathematical models, which is a Common Core Standard. Students can explore the idea of fractional dimension through observing and quantifying observations they make about natural and man-made phenomenon. Students in Algebra and Geometry must understand ideas involving proportions and scale that directly relate to the concept of self-similarity.

The wide range of applications involving fractals gives opportunities for teachers to enhance and widen the scope of their curriculum. One mathematician, Dr. Ron Eglash, designed Culturally Situated Design Tools (CSDT, csdt.rpi.edu), for teachers and students to engage with fractals. The tools do not overwhelm the students with the mathematics behind fractals but do provide teachers with tools needed to better explain the mathematical concepts to students. The java applets allow the teacher to better **differentiate** (meet the needs of all the students in the classroom) as they can explore the tools at their own speed and the teacher can generate extension activities based on the needs of the students.

One positive benefit of using the CSDT tools is that students make the connection between mathematics and the real world, often times, exploring their own culture or the cultures of their peers (Eglash, 222-223). This helps them challenge negative stereotypes they have regarding their own mathematics abilities as well as their peers, a powerful way to address issues of race and class in the mathematics classroom.

The African Fractals java applet is excellent example of meshing theoretical mathematics and making it accessible and relevant for high school students. The tool is divided into three sections, Background, Applications, and African Culture. In the Background section, students explore some fundamental definitions that make up fractal geometry such as recursion, fractal dimension, and iterate their own fractal design based on the Koch curve. In applications, students investigate and design fractals that come from natural occurring phenomena such as non-linear spirals from plants, ferns/algae, iterations with Da Vinci's tree drawing, crystal structures, engineering such as Sierpiński gasket antenna models for cell phones, and human structures such as lungs. Finally, in the African Fractals section, they explore background regarding fractals in African architecture, art, religion, and how they may be used in the future of their society. Students are able to design the fractals in-between the historical and cultural discussions, via the java applets (Eglash, "CSDT").

Conclusion

We have discussed the mathematical roots of fractal geometry, applications to the real world, and finally, ways of introducing secondary students to this topic. As more research is done utilizing fractals in both pure and applied settings, scientists, mathematicians, econophysicists, and computer scientists will find ways to better explain phenomena using this tool of mathematics. The future looks bright for fractals!

Bibliography

Barnsley, Michael. *Fractals Everywhere, Second Edition*. ed. Cambridge, MA: Academic Press, Inc., 1993. Print.

Barnsley, Michael, Dietmar Saupe, and Edward Vrscay. *Fractals in Multimedia*. NewYork: Springer-Verlag, 2002. Print.

Banu, S., J. Barba, and K.S. Chan. "Texture Analysis in Cytology using Fractals." *Applications of Digital Image Processing* 273: 1990. Web. 22 July 2014.<http://dx.doi.org/10.1117/12.962330>

Brown, William. "Mandelbrot set is as complicated as it could be." *New Scientist* 28 Sept. 1991: n.d. Web. 17 July 2014.<https://www.math.auckland.ac.nz/~king/745/fractdim.pdf>

Brennan, Will. "Fractals in Finance and Risk." Notre Dame University, n.d. Web. 22 July 2014. <http://www3.nd.edu/~malber/sem_un/Project%201/Fractals%20in%20Finance%20and %20Risk.ppt>.

Budinski-Petkovic, Lj., I. Loncarevic, Z.M. Jaksic, and S.B. Vrhovac. "Fractal properties of financial markets." *Physica A* 410: 43-53. *Fractal properties of financial markets*. Web. 22 July 2014. <http://www.sciencedirect.com/science/article/pii/S0378437114003811>

Crilly, A.J. , R.A. Earnshaw, and H. Jones. *Fractals and Chaos*. New York: Springer-Verlag, 1991. Print.

Devaney, Robert. *An Introduction to Chaotic Dynamical Systems*. ed. Boulder, CO: Westview Press, 2003. Print.

Devaney, Robert. “The Sierpirt. Triangle.” N.p., 2 Apr. 1995. Web. 10 July 2014. <http://math.bu.edu/DYSYS/chaos-game/node2.html>

Eglash, Ron. *African Fractals: Modern Computing and Indigenous Design*. Piscataway, NJ: Rutgers University Press, 1999. Print.

Eglash, Ron. “Culturally Situated Design Tools: TEACHING MATH AND COMPUTING THROUGH CULTURE.” Rensselaer Polytechnic Institute, 1 Jan. 2003. Web. 22 July 2014. <http://csdt.rpi.edu/>.

Frailey, Danielle. “Fractals.” University of Georgia , n.d. Web. 10 July 2014. <http://jwilson.coe.uga.edu/EMAT6680Fa11/Frailey/FractalEssay.htm>

Gould, Daniel, Tegy Vaddakan, Ross Poche, and Mary Dickinson. Multifractal and Lacunarity Analysis of Microvascular Morphology and Remodeling. NIH, 1 Feb. 2011. Web. 22 July 2014. <http://www.ncbi.nlm.nih.gov/pmc/ articles/PMC3049800/>.

Jacobsson, Emilie. “How to predict crashes in financial markets with the Log-Periodic Power Law.” Stockholm University, 1 Sept. 2009. Web. 22 July 2014.<http://www.chronostraders.com/wpcontent/uploads/2013/08/Research_on_LPPL.pdf.>

Johnson, Niles. “Point Set Toplogy.” . N.p., n.d. Web. 10 July 2014. <http://www.nilesjohnson.net/point-set-topology-topics.html>.

“Koch Curve.” Fractal Foundation, 1 Jan. 2013. Web. 10 July 2014. <http://fractalfoundation.org/resources/fractivities/koch-curve/>.

Lee, Y.K., and Kelvin Hoon. “Brownian Motion.” . N.p., 1 Jan. 1995. Web. 22 July 2014. <http://www.doc.ic.ac.uk/~nd/surprise_95/journal/ vol4/ ykl/report.html>.

Lipa, Chris. “Chaos and Fractals: The Mandelbrot Set.” Cornell University, n.d. Web. 17 July 2014. <http://www.math.cornell.edu/~lipa/mec/lesson5.html>.

Mandelbrot, B.B. . “How long is the coast of Britain? Statistical self-similarity and fractional dimension.” *Science* 156: 636-638. Web. 3 July 2014.

O'Connor, J.J., and E.F. Robertson. “Edward Norton Lorenz.”. School of Mathematics and Statistics University of St Andrews, Scotland , 1 July 2008. Web. 1 June 2014. <http://www-history.mcs.stand.ac.uk/ Biographies/Lorenz_Edward.html>.

Riddle, Larry. “Area of the Koch Snowflake.” . Larry Riddle, 1 May 2013. Web. 10 July 2014. <http://ecademy.agnesscott.edu/~lriddle/ifs/ksnow/area.htm>.

“Sequences and Series – Von Koch's Snowflake Curve.” . N.p., n.d. Web. 10 July 2014. <http://astarmathsandphysics.com/ib-maths-notes/sequences-and-series/ib-maths-notes-von-kochs-snowflake-curve.html>.

Shishikura, Mitsuhiro. “The Hausdorff Dimension of the Boundary of the Mandelbrot Set and Julia Sets.”*Annals of Mathematics* 142: 225-267. Web. 22 July 2014.<http://ftp.math.sunysb.edu/preprints/ims91-7.pdf>

Stewart, Ian, Arthur Clarke, Benoit Mandelbrot, Michael Barnsley, Louisa Barnsley, Will Rood, Gary Flake, David Pennock, and Nigel Lesmoir-Gordon. *The Colours of Infinity: The Beauty and Power of Fractals*. London: Springer-Verlag, 2010. Print.

Smith, T.G., G.D. Lange, and W.B. Marks. “Fractal methods and results in cellular morphology--dimensions, lacunarity and multifractals..” *Journal of Neuroscience Methods* 69: 123-136. Web. 22 July 2014. <http://www.sciencedirect.com/science/article/pii/S0165027096000805>